- assumptions:

$c_t : S \to \mathbb{R}$ $\quad c_t(x) \in [-C, C]$

$r\mathbb{B} \subseteq S \subset R\mathbb{B}$

c_t's are L-Lipshitz

$|c_t(x) - c_t(y)| \le L|x-y|$

Lecture 22

Alg:
- $y_1 = 0$
- at each period t
  - select uniform, random unit vector u_t
  - $x_t = y_t + \delta u_t$
  - "play" x_t & observe $c_t(x_t)$
  - $y_{t+1} = P_{(1-\alpha)S}\left(y_t - \nu c_t(x_t) u_t\right)$

- $$E_{u \in \mathbb{S}}\left[\frac{d}{\delta} f(x+\delta u)\, u\right] = \nabla \hat{f}(x)$$

$$\hat{f}(x) = E_{v \in \mathbb{B}}\left[f(x+\delta v)\right] \text{ - smoothed } f.$$

- $$E\left[\sum_{t=1}^{n} c_t(z_t)\right] - \min_{x \in S} \sum_{t=1}^{n} c_t(x) \le RG\sqrt{n}$$

where
- $z_1 = 0$, $z_{t+1} = P_S(z_t - \eta g_t)$
- $\eta = \frac{R}{G\sqrt{n}}$
- $E[g_t \mid z_t] = \nabla c_t(z_t)$
- $\|g_t\| \le G$
- $S \subseteq R\mathbb{B}$ ← radius of S.

"min in $(1-\alpha)S$ is close to min in S."

(*) **Lemma:** $\min_{x \in (1-\alpha)S} \sum_{t=1}^{n} c_t(x) \le 2\alpha C n + \min_{x \in S} \sum_{t=1}^{n} c_t(x)$

pf: • $x \in S \Rightarrow (1-\alpha)x \in (1-\alpha)S$

$$c_t((1-\alpha)x) \le (1-\alpha)c_t(x) + \alpha c_t(0)$$

$$\le c_t(x) + \underbrace{2\alpha C}_{\text{since } |c_t(x)|,\ |c_t(0)| \le C}$$

• Sum over n periods, subst x^* for x

$$x^* = \operatorname{argmin}_{x \in S} \sum_{t=1}^{n} c_t(x)$$

$$\min_{x \in (1-\alpha)S} \sum_{t=1}^{n} c_t(x) \le \sum_{t=1}^{n} c_t((1-\alpha)x^*) \le 2\alpha C n + \sum_{t=1}^{n} c_t(x^*)$$

(**) **Lemma:** $\forall x \in (1-\alpha)S$ the ball of radius αr centered at x is contained in S

pf: $S = (1-\alpha)S + \alpha S$

$\Rightarrow \forall x \in S \Rightarrow \exists y \in (1-\alpha)S,\ z \in \alpha S$ s.t.

$$x = y + z$$

$$y = (1-\alpha)x \qquad z = \alpha x.$$

$\Leftarrow \forall y \in (1-\alpha)S,\ z \in \alpha S \Rightarrow \exists x \in S$ s.t.

$$x = y + z.$$

$\hat{y} \in S$ s.t. $(1-\alpha)\hat{y} = y$ $\qquad (1-\alpha)\hat{y} + \alpha\hat{z} \in S$

$\hat{z} \in S$ s.t. $\alpha \hat{z} = z$

since S is convex.

$$S = (1-\alpha)S + \alpha S \supseteq (1-\alpha)S + \alpha r \mathbb{B}$$

since $r\mathbb{B} \subseteq S$.

<u>Thm</u> for $n \geq \left(\frac{3Rd}{2r}\right)^2$ and for $\nu = \frac{R}{C\sqrt{n}}$

$\delta = \sqrt[3]{\frac{rR^2d^2}{12n}}$ $\qquad \alpha = \sqrt[3]{\frac{3Rd}{2r\sqrt{n}}}$

the expected regret of Alg is bounded by.

<u>Thm:</u> If each c_t is L-Lipshitz, for n sufficiently large and $\nu = \frac{R}{C\sqrt{n}}$, $\alpha = \frac{\delta}{r}$

and $\delta = n^{-\frac{1}{4}}\sqrt{\frac{RdCr}{3(Lr+C)}}$

then expected regret of Alg is bounded by

$$2n^{3/4}\sqrt{3RdC\left(L + C/r\right)}.$$

<u>pf.</u> • begin by showing each $x_t \in S$.

• $y_t \in (1-\alpha)S$ by lemma (**) know that

• if $\delta \leq \alpha r$ we are good.

• we set $\alpha = \frac{\delta}{r} \Rightarrow \delta = \alpha r$.

- Let $\hat{c}_t(x)$ be the smoothed version of $c_t(x)$.

  - we firsth show a good regret result using the $\hat{c}$'s y_t's then we show that using the original c's & x's can't hurt the result too much.

- Let $\hat{c}_t(x) = E_{v \in \mathbb{B}}\left[c_t(x + \delta v) \right]$

- Let the feas region be $(1-\alpha)S$.

- Let $g_t = \frac{d}{\delta} c_t(y_t + \delta u_t) u_t$.

- By estimate lemma. $E[g_t \mid y_t] = \nabla \hat{c}_t(y_t)$

- So we can apply exp. gradient descent rules:

$$y_{t+1} = P_{(1-\alpha)S}\left(y_t - \eta g_t\right)$$

$$y_{t+1} = P_{(1-\alpha)S}\left(y_t - \underbrace{\eta \frac{d}{\delta}}_{\text{step size} - \nu} c_t(y_t + \delta u_t) u_t\right)$$

but otherwise, same rule as we use

$$\|g_t\| = \left\|\frac{d}{\delta} c_t(y_t + \delta u_t) u_t\right\| \le \frac{d}{\delta} C = "G"$$

$$\frac{R}{C\sqrt{n}} = \nu = \eta \frac{d}{\delta} \Rightarrow \eta = \frac{\delta R}{dC\sqrt{n}} = \frac{R}{G\sqrt{n}}$$

• so, exp grad lemma tells us.

$$E\left[\sum_{t=1}^{n} \hat{c}_t(y_t)\right] - \min_{x \in (1-\alpha)S} \sum_{t=1}^{n} \hat{c}_t(x) \le RG\sqrt{n}$$

$$= \frac{RdC}{\delta}\sqrt{n}$$

◦ ok, so now we have to get rid of the $\hat{c}$'s y_t's. $(1-\alpha)S$.

◦ Let's think about switching from $\hat{c}$ to c.

$$|\hat{c}_t(x) - c_t(x)| = \left| \int_v \rho(v)\, c_t(x+\delta v) - \int_v \rho(v)\, c_t(x) \right|$$

$$= \left| \int_v \rho(v) \left[c_t(x+\delta v) - c_t(x) \right] \right|$$

$$\le \int_v \rho(v) \left| c_t(x+\delta v) - c_t(x) \right|$$

$$\le \int_v \rho(v)\, \delta L$$ ← since c_t's are L-Lipschitz.

$$= \delta L$$

$$|\hat{c}_t(x) - c_t(x)| \le \delta L$$

• but we also want to go from y's to x's.

$$|\hat{c}_t(y_t) - c_t(x_t)| \le |\hat{c}_t(y_t) + c_t(y_t) - c_t(y_t) - c_t(x_t)|$$

$$\le |\hat{c}_t(y_t) - c_t(y_t)| + |c_t(y_t) - c_t(x_t)|$$

$$\le \delta L + \delta L \quad \text{, since } x_t = y_t + \delta u_t$$

$$= 2\delta L$$

• great! all we have left is the $(1-\alpha)S$.

$$E\left[\sum_{t=1}^{n} \left(c_t(x_t) - 2\delta L\right)\right] - \min_{x\in(1-\alpha)S} \sum_{t=1}^{n} \left(c_t(x) + \delta L\right)$$

$$\le \frac{RdC\sqrt{n}}{\delta}$$

but we know that the mins are close by Lemma (*), so

$$E\left[\sum_{t=1}^{n} c_t(x_t)\right] - \min_{x\in S} \sum_{t=1}^{n} c_t(x) - 2\alpha Cn$$

$$\le \frac{RdC\sqrt{n}}{\delta} + 3\delta L n$$

or regret bnd by

$$\frac{RdC\sqrt{n}}{\delta} + 2\alpha Cn + 3\delta Ln$$

$$= \frac{RdC\sqrt{n}}{\delta} + \frac{2\delta Cn}{r} + 3\delta Ln$$

plugin $\delta \propto n^{-\frac{1}{4}}$ gives the regret of $n^{3/4}$. $\blacksquare$