# OA 4202, Homework 2

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Due Mon Oct. 11 at the beginning of class

1. Drawing on your past personal experiences, give an example application for finding shortest paths.

2. Run Dijkstra's algorithm on the graph in Figure 1 with start node 0. Specifically,

   (a) List the order in which nodes are popped off the priority queue.

   (b) For each node $u$, list the sequence of values given to $\tilde{d}(s, u)$.

   (c) Draw the final search tree of the algorithm.

3. Give a small example of a graph with one negative length edge where there are no negative length cycles but Dijkstra's algorithm doesn't work. Give a short explanation about why the algorithm fails to work in your example.

4. (AMO)

   **5.21.** Professor May B. Wright suggests the following method for solving the shortest path problem with arbitrary arc lengths. Let $c_{\min} = \min\{c_{ij} : (i, j) \in A\}$. If $c_{\min} < 0$, add $|c_{\min}|$ to the length each arc in the network so that they all become nonnegative. Then use Dijkstra's algorithm to solve the shortest path problem. Professor Wright claims that the optimal solution of the transformed problem is also an optimal solution of the original problem. [redacted]

   Give a small example showing that the claim is false.

5. (AMO)

   **4.5. Money-changing problem.** The money-changing problem requires that we determine whether we can change a given number $p$ into coins of known denominations $a_1$, $a_2, \ldots, a_k$. For example, if $k = 3$, $a_1 = 3$, $a_2 = 5$, $a_3 = 7$, we can change all the numbers in the set $\{8, 12, 54\}$; on the other hand, we cannot change the number 4. In general, the money-changing problem asks whether $p = \sum_{i=1}^{k} a_i x_i$ for some nonnegative integers $x_1, x_2, \ldots, x_k$.

   **(a)** Describe a method for identifying all numbers in a given range of numbers $[l, u]$ that we can change.

   **(b)** Describe a method that identifies whether we can change a given number $p$, and if so, then identifies a denomination with the least number of coins.

   (c) For a given number $p$, describe a method that counts the number of ways that $p$ can be changed.

6. (AMO)

**4.13. Single-duty crew scheduling.** The following table illustrates a number of possible duties for the drivers of a bus company. We wish to ensure, at the lowest possible cost, that at least one driver is on duty for each hour of the planning period (9 A.M. to 5 P.M.). Formulate [redacted] this scheduling problem as a shortest path problem.

| Duty hours | 9–1 | 9–11 | 12–3 | 12–5 | 2–5 | 1–4 | 4–5 |
|---|---|---|---|---|---|---|---|
| Cost | 30 | 18 | 21 | 38 | 20 | 22 | 9 |

7. (AMO)

**4.2.** Beverly owns a vacation home in Cape Cod that she wishes to rent for the period May 1 to August 31. She has solicited a number of bids, each having the following form: the day the rental starts (a rental day starts at 3 P.M.), the day the rental ends (checkout time is noon), and the total amount of the bid (in dollars). Beverly wants to identify a selection of bids that would maximize her total revenue. Can you help her find the best bids to accept?

8. (AMO)

**4.9. Personnel planning problem** (Clark and Hastings [1977]). A construction company's work schedule on a certain site requires the following number of skilled personnel, called *steel erectors*, in the months of March through August:

| Month | Mar. | Apr. | May | June | July | Aug. |
|---|---|---|---|---|---|---|
| Personnel | 4 | 6 | 7 | 4 | 6 | 2 |

Personnel work at the site on the monthly basis. Suppose that three steel erectors are on the site in February and three steel erectors must be on site in September. The problem is to determine how many workers to have on site in each month in order to minimize costs, subject to the following conditions:

*Transfer costs.* Adding a worker to this site costs \$100 per worker and redeploying a worker to another site costs \$160.

*Transfer rules.* The company can transfer no more than three workers at the start of any month, and under a union agreement, it can redeploy no more than one-third of the current workers in any trade from a site at the end of any month.

*Shortage time and overtime.* The company incurs a cost of \$200 per worker per month for having a surplus of steel erectors on site and a cost of \$200 per worker per month for having a shortage of workers at the site [redacted]

Formulate this problem as a shortest path problem [redacted] (*Hint*: [redacted] use as many nodes for each month as the maximum possible number of steel erectors.)

9. You are in a foreign country where terrorists occasionally attack road convoys. You know the country's transportation network as a graph where roads (edges) connect cities (nodes). For every road, based on historical data, you know the probability that terrorists will attack a convoy traversing the road. For example, if there is a road between city $i$ and city $j$, you know the probability an attack will occur on the road, $p_{ij}$. Attacks on each of the roads occur independently of one another.

   You have to move a convoy from city $A$ to city $B$, using the country's road network. Describe how to find the safest, most reliable path connecting city $A$ and city $B$. (Hint: Define the "reliability" of a path as the probability that a convoy using the path will not be attacked).

10. (This is a bonus problem. It is not part of the required homework because it is not central to the class material. However, it is a good puzzle problem, and if you do the problem correctly, you'll receive +7% extra on this homework assignment.) In class, we mentioned that a heap data structure gives us the following operations and run times:

| name | description | run time |
|---|---|---|
| find_min | Look at the element with the minimum key value | $O(1)$ |
| delete_min | Delete the element with the minimum key value | $O(\log n)$ |
| insert | Insert an element with a given key value | $O(\log n)$ |
| decrease_key | Decrease the key value of a specified element | $O(\log n)$ |

Describe how those operations allow us to also do "increase_key" in $O(\log n)$ time.

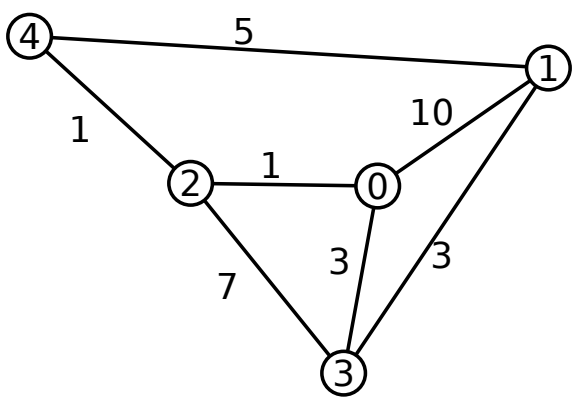


Figure 1: An example graph.