Notes 3.1

There are generally 4 types of network models that we'll study in class, and one "hybrid" between these. Each is applicable to many real-world situations

- You are already familiar with one:

Shortest Path

$$\min_y \sum c_{ij} y_{ij}$$

$$\text{s.t.} \quad \sum y_{is} - \sum y_{si} = -1$$

$$\sum y_{it} - \sum y_{ti} = 1$$

$$\sum y_{ia} - \sum y_{ai} = 0 \quad \forall a \neq s, \ a \neq t$$

$$0 \leq y_{ij}$$

the data for this problem are some kind of edge "lengths", c_{ij}

- the objective function is to minimize the total cost of a path
- and the y_{ij} variables give a path (1 on the path, 0 elsewhere)

- Sometimes, a twist on the shortest path problem is if we have some extra side constraint.

- for example, suppose it costs f_{ij} fuel to move across edge ij, and we have a total budget of b for fuel.

we can add

$$\sum f_{ij} y_{ij} \leq b$$

as a constraint

This is a useful model, but it breaks things like unimodularity of the constraint matrix ... more on this later in the course

Max Flow

- suppose now we have capacities on each edge instead of lengths.
- let u_{ij} be the capacity of edge ij
- now, we want to see how many trucks we can move from s to t.

$$\max_{v,y} \quad v$$

s.t.
$$\sum y_{is} - \sum y_{si} = -v$$
$$\sum y_{it} - \sum y_{ti} = v$$
$$\sum y_{ia} - \sum y_{ai} = 0 \quad \forall a \neq s,\ a \neq t$$
$$y_{ij} \leq u_{ij}$$
$$0 \leq y_{ij}$$

- the data for this problem are arc capacities, u_{ij}
- the objective function is to maximize the total flow from s to t
- the y_{ij} variables now give the # of trucks moving on arc ij.

- there are many twists to this problem, but a common one is to also have lower bounds on the edges

$$l_{ij} \leq y_{ij}$$

this requires at least l_{ij} trucks to go through edge (i,j)

Min - Cost Flow

- in this problem, we have

  c_{ij} - cost for driving a truck on edge ij

  u_{ij} - capacity of edge ij

  $b(a)$ - supply or demand of material at node a.
  negative for supply
  positive for demand
  zero for trans-shipment nodes.

- the goal is to move all the supplys to all the demands at minimum cost, while satisfying the capacity constraints

$$\min_{y} \quad \sum c_{ij} y_{ij}$$

$$\text{s.t.} \quad \sum y_{ia} - \sum y_{ai} = b(a)$$

$$y_{ij} \leq u_{ij}$$

$$0 \leq y_{ij}$$

- the data you need here are the c_{ij}, u_{ij} and $b(a)$ (see above)
- a common twist here is also lower bounds on the y_{ij}.
- y_{ij} variables give # of trucks on arc (i,j)

Multi-commodity flow

- Now, suppose that there are different kinds of demands... some for corn, some wheat, some rice.

c_{ij}^k - cost for moving one unit of commodity k across edge (i,j)

u_{ij}^k - capacity of moving commodity k across edge (i,j)

$b^k(a)$ - supply/demand of commodity k on node a.

u_{ij} - common capacity for the sum of all commodities moving across edge ij.

$$\min_y \sum_k \sum_{ij} c_{ij}^k y_{ij}^k$$

s.t.

$$\sum y_{ia}^k - \sum y_{ai}^k = b(a) \quad \forall a \in V, \text{ commodities } k.$$

$$y_{ij}^k \leq u_{ij}^k \quad \forall (i,j) \in E, \text{ commodities } k$$

$$\sum_k y_{ij}^k \leq u_{ij} \quad \forall (i,j) \in E$$

$$0 \leq y_{ij}^k$$

- here the y_{ij}^k variables give the # of trucks moving across edge ij that carry commodity k. e.g. y_{ij}^{rice}

Stochastic Network Model (Hybrid)

- You can combine these in many different ways
- Multi-commodity flow itself is a combination of min-cost flows, one for each commodity.
- "Combination" just involves picking weights that signify importance for two different problems, and summing their weighted objective function
- For example... lets say I don't know if I want the shortest path from s to t or from $\hat{s}$ to $\hat{t}$, but I am $2/3$ sure it is s to t.
- we can create a combined problem as follows.

$$\min \; \frac{2}{3} \sum c_{ij}\, y^1_{ij} \; + \; \frac{1}{3} \sum c_{ij}\, y^2_{ij}$$

s.t.

$$\sum y^1_{is} - \sum y^1_{si} = -1 \qquad \sum y^2_{i\hat{s}} - \sum y^2_{\hat{s}i} = -1$$

$$\sum y^1_{it} - \sum y^1_{ti} = 1 \qquad \sum y^2_{i\hat{t}} - \sum y^2_{\hat{t}i} = 1$$

$$\sum y^1_{ia} - \sum y^1_{ai} = 0 \qquad \sum y^2_{ia} - \sum y^2_{ai} = 0$$

$$0 \le y^1_{ij} \qquad 0 \le y^2_{ij}$$

totally separate network problems, but weighted and summed together.

- in fact, the two problems can have entirely different network structure or costs!

it is called "stochastic" because the weights are often from a probability distribution on real-world situations

e.g. high traffic days vs. low traffic days

- can similarly combine shortest paths and max-flows or any combo of 2 or more of the basic models
- can add cross-constraints like: $y^1_{ij} + y^2_{ij} \le 1$ but often, these break nice properties.