OA 4202, Homework 2 Solutions

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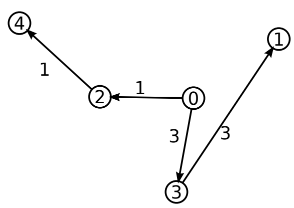
1. Run Dijkstra's algorithm on the graph in Figure 1 with start node 0. Specifically,
   (a) List the order in which nodes are popped off the priority queue.
   (b) For each node u, list the sequence of values given to $\tilde{d}(s, u)$.
   (c) Draw the final search tree of the algorithm.

   An acceptable solution:

   (a) $0, 2, 4, 3, 1$
   (b) $\tilde{d}(0,0) = 0$, $\tilde{d}(0,2) = 1$, $\tilde{d}(0,4) = 2$, $\tilde{d}(0,3) = 3$, $\tilde{d}(0,1) = 10, 7, 6$
   (c) 

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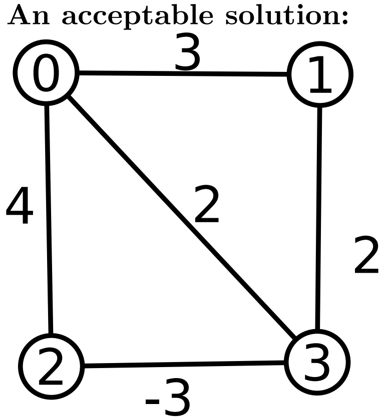
2. Give a small example of a graph with one negative length edge where there are no negative length cycles but Dijkstra's algorithm doesn't work. Give a short explanation about why the algorithm fails to work in your example.

   An acceptable solution:



   with a start node of 0.

   Once node 3 is popped off the priority queue, its distance is fixed to 2, and the algorithm continues from there. Before node 3 is popped off the priority queue, the algorithm has no opportunity to see and account for the path 0,2,3, which gives a distance of 1 from node 0 to node 3.

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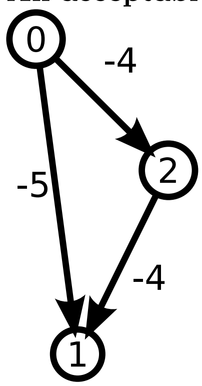
3. (AMO)

5.21. Professor May B. Wright suggests the following method for solving the shortest path problem with arbitrary arc lengths. Let $c_{\min} = \min\{c_{ij} : (i, j) \in A\}$. If $c_{\min} < 0$, add $| c_{\min} |$ to the length each arc in the network so that they all become nonnegative. Then use Dijkstra's algorithm to solve the shortest path problem. Professor Wright claims that the optimal solution of the transformed problem is also an optimal solution of the original problem. [redacted]

Give a small example showing that the claim is false.

An acceptable solution:



In this graph, the shortest path between 0 and 1 is $0, 2, 1$ with a length of -8. But, adding 5 to all edges to get a graph with non-negative length edges, gives that path a total length of 2, while giving the path $0, 1$ a total length of 0.

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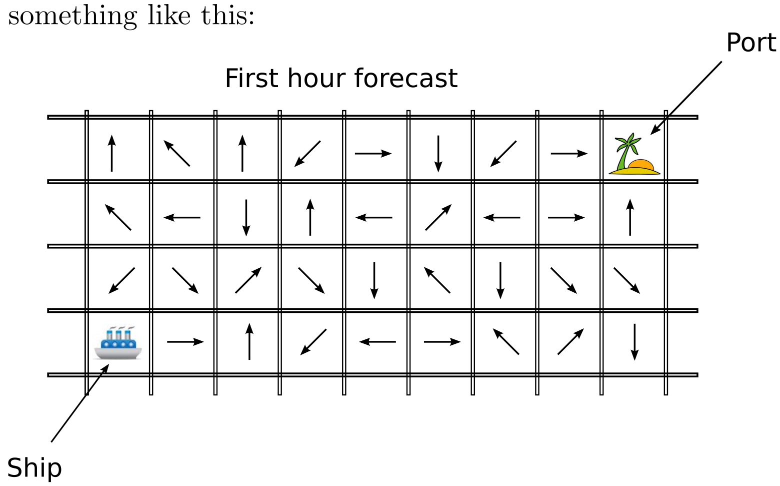
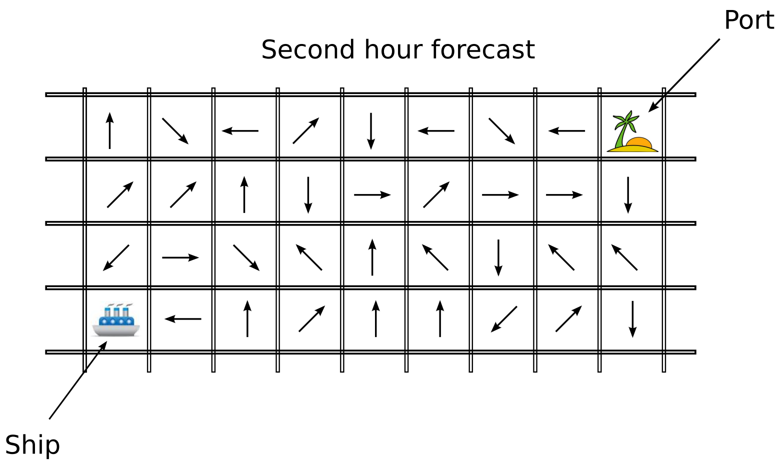
4. Imagine you are navigating a ship, and you would like to get to port by using the least amount of fuel. You have a weather forecast, that gives you the direction and magnitude of the waves for each small grid square of ocean. For our problem we'll imagine that magnitude is the same everywhere, so that only the direction matters. Then, an imaginary forecast might look something like this:



We have the forecast for the next three hours, split up in one hour time blocks. So, if the picture above was the forecast for the first hour, for the second hour, conditions may change to look like this:



Suppose that we also know the following fuel usage characteristics:

Wave direction	Fuel Usage per grid square
Same as direction of ship	0.5 units
45 degrees, but same direction as ship	1 unit
Perpendicular to direction of ship	2 unit
45 degrees, but opposite direction of ship	2.5 unit
Opposite direction of ship	3 units

We also know that it takes 15 minutes to traverse each grid square, regardless of the wave direction.

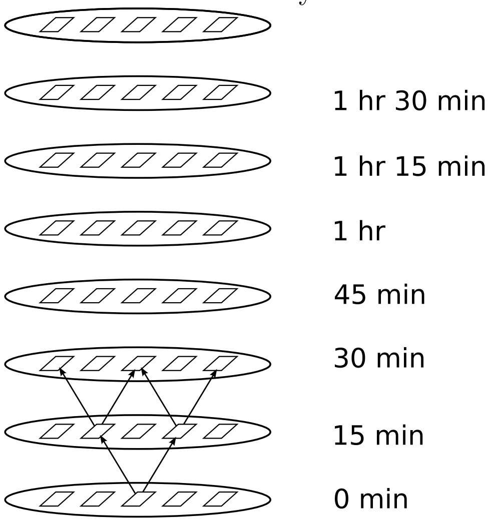
Formulate the problem of getting to port using the least amount of fuel as a shortest path problem. (Hint: First, how would you solve the problem if the forecast/wave conditions stay constant for all time? Once you have a solution for that, how would you incorporate a changing forecast?)

An acceptable solution:
To solve the problem when the forecast stays constant we can create a graph where each node is the center of a grid square, and each square is connected to its 8 neighbors with a directed edge.

To compute the edge costs, in units of fuel, we can compute the angle between the edge direction and the wave direction from the forecast. We can make this computation quite detailed. For example, an edge between two squares lies half in one square and half in the other, making half of the fuel usage come from the first square's angle and half from the second square's angle. The distance between squares that don't share an edge is $\sqrt{2}$ larger than the distance between squares that do share an edge, and we could make those fuel costs proportionally larger.

The shortest path from the ship position to port, would then give us the smallest fuel usage route.

When the forecast is changing, we can layer the network, creating one layer for each time period. So the bottom layer, would designate time 0, the next one 15 minutes, the next one 30 minutes and so forth. Since it takes 15 minutes to traverse from one grid square the edges would connect squares in layer 0 to squares in layer 15 minutes. The edge costs can be computed in a similar way as when the forecast doesn't change, but since we have a different layer for each time period, we can now account for changing wave directions. Here is an illustration of a time-layered network:



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5. (AMO)

4.2. Beverly owns a vacation home in Cape Cod that she wishes to rent for the period May 1 to August 31. She has solicited a number of bids, each having the following form: the day the rental starts (a rental day starts at 3 P.M.), the day the rental ends (checkout time is noon), and the total amount of the bid (in dollars). Beverly wants to identify a selection of bids that would maximize her total revenue. Can you help her find the best bids to accept?

An acceptable solution:

Create a graph in the following way. Each day between May 1 and Aug 31 is a node. Connect each day to the following day with a directed edge of length 0. If there is a bid for the time from day A to day B with dollar value k, then place a directed edge from node A to node B and give it length $-k$.

This creates a directed acyclic graph. In addition, the shortest path (most negative length path) from May 1 to Aug 31 gives us the bids to accept to maximize revenue.

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6. (AMO)

4.9. Personnel planning problem (Clark and Hastings [1977]). A construction company's work schedule on a certain site requires the following number of skilled personnel, called *steel erectors*, in the months of March through August:

Month	Mar.	Apr.	May	June	July	Aug.
Personnel	4	6	7	4	6	2

Personnel work at the site on the monthly basis. Suppose that three steel erectors are on the site in February and three steel erectors must be on site in September. The problem is to determine how many workers to have on site in each month in order to minimize costs, subject to the following conditions:

Transfer costs. Adding a worker to this site costs $100 per worker and redeploying a worker to another site costs $160.

Transfer rules. The company can transfer no more than three workers at the start of any month, and under a union agreement, it can redeploy no more than one-third of the current workers in any trade from a site at the end of any month.

Shortage time and overtime. The company incurs a cost of $200 per worker per month for having a surplus of steel erectors on site and a cost of $200 per worker per month for having a shortage of workers at the site [redacted]

Formulate this problem as a shortest path problem [redacted] (*Hint*: [redacted] use as many nodes for each month as the maximum possible number of steel erectors.)

An acceptable solution:

Create the following graph. The nodes will be pairs of (month, number of workers). We have one "start" node (February, 3 workers), and one "end" node (September, 3 workers). For each month, March through August, we create nodes for any number of workers from 0 to 21. . . the number 21 comes from the maximum number of workers we could ever have, since we start with 3 and can add at most 3 every month.

We can now add edges with lengths according to the costs specified in the problem. For example, we can add an edge from (February, 3 workers) to (March, 4 workers) with a cost of $100, because it costs $100 to transfer one worker into the site, and there are no additional penalties in terms of shortage time or over time. We also add an edge from (February, 3 workers) to (March, 5 workers) with a cost of $400, since it costs 200 to transfer in two people, and there is a 200 for having one extra worker on the site in March than what is needed.

After adding all the appropriate edges, the shortest path from (February, 3 workers) to (September, 3 workers) specifies a lowest cost plan for running the construction site.

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7. You are in a foreign country where terrorists occasionally attack road convoys. You know the country's transportation network as a graph where roads (edges) connect cities (nodes).

For every road, based on historical data, you know the probability that terrorists will attack a convoy traversing the road. For example, if there is a road between city i and city j, you know the probability an attack will occur on the road, p_{ij}. Attacks on each of the roads occur independently of one another.

You have to move a convoy from city A to city B, using the country's road network. Describe how to find the safest, most reliable path connecting city A and city B. (Hint: Define the "reliability" of a path as the probability that a convoy using the path will not be attacked).

An acceptable solution:
Recall that $\log(ab) = \log(a) + \log(b)$.

For each edge (i, j), the probability that no attack will occur on the edge is $1 - p_{ij}$. Define reliability of a path P is just the probability that the convoy will make it through without any attack: $\prod_{e \in P}(1 - p_e)$.

We would like to find the most reliable path between A and B. In other words, we would like to find the path, P^*, from A to B that maximizes the objective function $\prod_{e \in P}(1 - p_e)$.

Take the log of the objective function to get $\sum_{e \in P} \log(1 - p_e)$. Since, log is a monotonic function (strictly increasing), maximizing $\prod_{e \in P}(1-p_e)$ is the same as maximizing $\sum_{e \in P} \log(1-p_e)$. Maximizing $\sum_{e \in P} \log(1 - p_e)$ is the same as minimizing $\sum_{e \in P} - \log(1 - p_e)$. Furthermore, since each p_e is between 0 and 1, the value of $1 - p_e$ is at most 1, and the value of $-\log(1 - p_e)$ is always non-negative.

So, we can solve the problem by giving each edge in the graph length $-\log(1-p_e)$ and running Dijkstra's algorithm to solve a shortest path problem between A and B.

Punchline: Max reliability path problems are just shortest path problems, where we do a $-\log$ transformation to get the edge lengths. We could have phrased the question in lots of different ways. For example, a common way is to say that each edge represents a communication channel and the p_e's are the probability that a message in the channel is not delivered.

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8. Create a homework problem on applications of shortest path or its algorithms. The problem you create shouldn't simply be computational, like "run Dijkstra on graph G," but instead something that cleverly applies or tests understanding of sortest path modeling or algorithms.

   An acceptable solution:
   You saw a few examples above of such homework problems. I am looking forward to reading the ones you created.

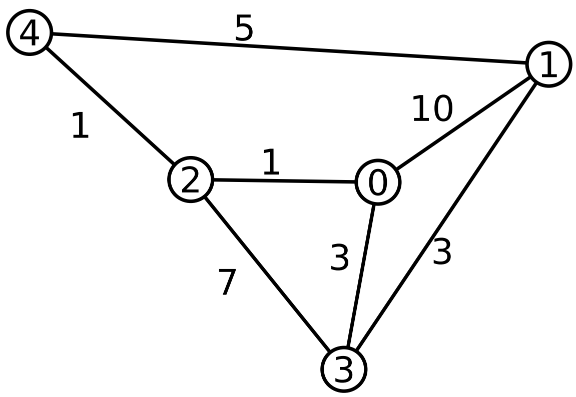

Figure 1: An example graph.