Optimal Feedback Allocation Algorithms For Multi-user Uplink

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***Abstract*—This paper investigates the impact of limited feedback on user throughput in the uplink of a cellular system. We consider scenarios where the base-station has limited feedback resources, which it needs to allocate across the users it serves. We propose a general model that captures the effect of feedback allocation on the achievable rates for a user, which allows us to characterize the rate region for such a system. For unsaturated queueing systems, we show that the optimal feedback allocation policy that stabilizes the queues when possible, involves solving a weighted sum-rate maximization at each scheduling instant. We show that such an online weighted sum-rate maximization policy can also be used for long-term utility maximization, which is applicable to saturated queueing systems. The weighted sum-rate maximization is solved using dynamic programming incurring pseudo-polynomial complexity in the number of users and in the total feedback bit budget. Finally, we show that the widely-studied single-stream multiple-input-multiple-output beamforming/combining physical layer communication strategy induces a special form on the optimal feedback allocation problem, which allows for the development of a polynomial-time approximation algorithm.**

I. Introduction

In many currently implemented wireless standards, channel state information (CSI) is fed back by the receiver to the transmitter to allow for the latter to adapt its transmit strategy. This includes power and rate adaptation, which is known to increase capacity over the case when there is no CSI at the transmitter (CSIT) and precoder adaptation in the case of multiple-input-multiple-output (MIMO) systems, which can be used to increase link reliability. Current state-of-the-art opportunistic scheduling algorithms such as multi-user diversity and proportional fairness assume the availability of CSIT through feedback, thus allowing for the transmitters to adapt their respective transmission strategies as a function of their link quality and other network state information. Consider multi-user diversity downlink scheduling for instance; the user with the best channel is scheduled in each time slot and the base station transmits (ideally) at the Shannon capacity of its link to that user. It is well-known (Sharif and Hassibi [1]) that for this scheduling policy, the sum-rate scales as $\Omega(\log \log K)^1$, where K is the number of users. However, as noted by Huang et al. [2] this increase comes with a linear increase in feedback rate. This observation has motivated the development of limited feedback techniques (see, e.g., [3], [5] and references therein).

Past literature on limited feedback can be broadly classified into two categories. The impact of limited feedback on the performance of MIMO point-to-point wireless links has been studied by Mondal et al. and Love at al. [3], [4]. A parallel body of work [5], [6] focuses on developing limited feedback protocols for multi-user orthogonal frequency-division multiple-access (OFDMA) downlink scheduling. Chen et al. [5] propose a limited feedback scheme where each user, with associated priority, is restricted to a feedback budget of one bit per tone, i.e., each user transmits a bit that indicates whether its channel is above a certain threshold. Given a set of users with good channels, the base station schedules the user with the highest priority on each tone. The authors compute thresholds that achieve the optimal trade-off between feedback rate and data rate for this class of data and feedback scheduling policies. While the above work assumes that the feedback window has number of slots equal to the product of the number of users and tones, Agarwal et al. [6] relax this assumption by considering feedback windows of arbitrary size. They propose an opportunistic feedback scheme where a user contends for a feedback slot if their channel strength is greater than a pre-set threshold.

In contrast to the aforementioned literature, in this paper, we investigate the impact of limited feedback on user throughput in the uplink of a cellular system. Explicit feedback for the uplink is required for current and future standards (such as Long Term Evolution) that employ frequency-division-duplexing (FDD) since channel reciprocity cannot be exploited for such systems. Fig. 1 depicts the uplink of a FDD cellular network where the base station serves multiple mobiles or users and has a limited feedback budget to allocate across these users. Specifically, we assume that the base station is constrained in the total number of feedback bits it can allocate across these users. Feedback allocation is necessary because limited feedback induces errors that

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[1] $f(n) = \mathcal{O}(g(n))$ if $\exists \bar{n}$ and $c_1 > 0$ such that $f(n) \leq c_1 g(n),\ \forall n \geq \bar{n}$; $f(n) = \Omega(g(n))$ if $f(n) = \mathcal{O}(g(n))$ and $\exists \bar{n}$ and $c_2 > 0$ such that $f(n) \geq c_2 g(n),\ \forall n \geq \bar{n}$.

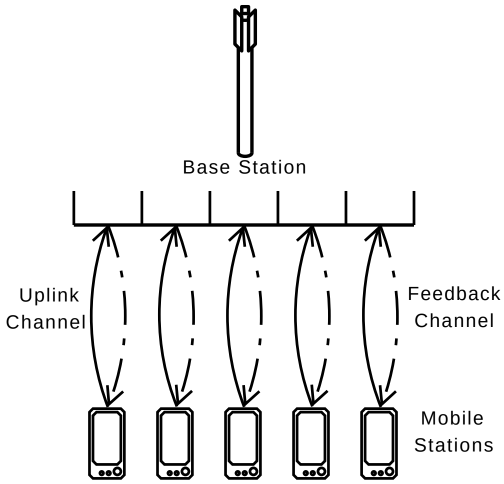

Fig. 1. FDD cellular uplink where the base-station has a feedback link to each user.

predominantly stem from quantization and delay. [2] We restrict our attention to an intuitively appealing, but nonetheless broad, space of feedback allocation policies in the interest of analytical tractability and implementability. The policies can be described as a *partitioning* of the total feedback bit budget across users. Thus, for the uplink scenario under consideration, if the network objective is fairness across users for instance, then a user with a poor channel would most likely be allocated a larger fraction of bits. On the contrary, if the objective is sum-rate maximization, a stronger user might be allocated a larger fraction of bits. More importantly, as a consequence of the total feedback constraint and independent of the choice of objective, the uncertainties in CSIT become coupled across the users, a fact that has not been explicitly modeled in past literature. The main contributions of this paper are the following:

1) We propose a limited feedback framework for cellular uplink that models this coupling in throughput performance across users.
2) An optimal multi-user feedback scheduling policy is presented, where we design this policy to achieve one of two long-term network objectives.
    a) Queue stability: This classical network objective [7], [8] is applicable to queueing systems where each user does not have infinitely back-logged data to transmit, henceforth referred to as *unsaturated* systems.
    b) Utility maximization: This second objective applies to systems that have infinitely back-logged data, called *saturated* systems [9].
3) We show that the optimal allocation can be computed using dynamic programming incurring pseudo-polynomial complexity in the number of users and in the total feedback bit budget.
4) For specific uplink deployments that employ single-stream multiple-input-multiple-output beamforming and combining, we show that the optimal feedback allocation problem takes a special form that allows for the development of a polynomial-time relaxation with associated approximation guarantees. Single-stream multiple-input-multiple-output beamforming and combining is being considered as a potential transmission mode in the Long Term Evolution standard [10].

[2]Quantization error is encountered during the process of estimating the channel at the receiver and mapping it to a set of bits or states in order to be sent back to the transmitter. Delay error is due to the fact that the signal passing through the feedback channel is received at the transmitter after some delay depending on the user's location and the fact that the true channel might have changed over this period.

The rest of this paper is organized as follows. In Section II, we introduce the system model for multi-user feedback scheduling. In Section III, we discuss the two long-term objectives that drive our choice of scheduling policies. We present a linear-optimization-based approach to compute throughput-optimal feedback allocations, and also provide a result useful later when we obtain approximate but computationally more friendly feedback allocation schemes. In Section IV, we present the optimal feedback allocation policy for both objectives while in Section V, we investigate methods of reducing the complexity of the optimal feedback allocation policy.

Notation: We introduce some notation for the sake of readability. x_{ij} denotes element (i,j) of matrix $\mathbf{X}$ while x_i denotes element i of vector $\mathbf{x}$. Given matrices $\mathbf{X},\mathbf{Y} \in \mathbb{R}^{p\times q}$, $X \leq Y$ means $x_{ij} \leq y_{ij}, \forall i = 1,\ldots,p,\ j = 1,\ldots,q$. $(.)^T$ and $(.)^\dagger$ are the transpose and Hermitian-transpose operators respectively. The sets $\mathbb{R}_+$, $\mathbb{N}_0$ and $\mathbb{N}$ represent the non-negative real numbers , non-negative integers and positive integers respectively. Finally, $[x]^+ = \max\{x,0\}$ and $||.||$ is the two-norm operator.

II. System model

Consider the uplink of a slotted-time cellular system with K users scattered across a cell. Each user-base-station channel is modeled as a finite-state discrete-time process where the composite channel across users (in appropriate units) at time t, $\mathbf{m}[t]$, takes values in set $\mathcal{M}, |\mathcal{M}| = M$. For example, if we model all the channels as Gilbert-Eliot (or ON-OFF channels), then $\mathcal{M} = \{0,1\}^K$. We assume that the base-station has perfect knowledge of the channel state $\mathbf{m}[t]$ in every time slot. Each user transmits on a separate frequency band thereby removing the need for data scheduling since the focus of this work is primarily on feedback scheduling. To this effect, we assume that the base station has an error-free control channel that is broadcast in nature, which it uses for feedback purposes. Each feedback packet has a total size B bits and is intended to carry quantized channel state information back to all users. The base station has to allocate $b_k, k = 1,\ldots,K$, bits of each feedback packet to user k such that $\sum_{k=1}^K b_k \leq B$. Let $\mathcal{B} = \{\mathbf{b} \in \mathbb{N}_0^K : \sum_{k=1}^K b_k \leq B,\ B \in \mathbb{N}\}$ represent the set of allowable bit allocation vectors. In each time slot, the base station decides on a bit allocation that it will use to form the feedback packet. An insufficiently large budget B will lead to loss of information in the quantization process. In addition to quantization effects, we assume the presence of delay in the feedback link, the combination of which motivates the following general transmission model. In channel state $\mathbf{m} \in \mathcal{M}$, user k chooses their transmission rate $\mu_k(m_k, b_k) \in \mathbb{R}_+$ based on:

- the bit allocation b_k
- the quantized CSIT that it receives
- its inherent tendency towards tolerating outage or packet drops

Since we assume that maximum tolerable outage probability remains fixed over the entire period that the user is in the system, we do not explicitly include it in the functional definition of rate $\mu_k(m_k, b_k)$. The above setup accurately models scenarios where:

1) The channel process $\{\mathbf{m}[t]\}$ is an ergodic Markov chain with a strictly positive feedback delay.
2) We have a zero-delay feedback link and the channel process $\{\mathbf{m}[t]\}$ is either independent and identically distributed (i.i.d.) across time or ergodic Markov.

We denote the stationary distribution (unique in the case of ergodic Markov) of the channel as $\{\pi_{\mathbf{m}}\}_{\mathbf{m}\in\mathcal{M}}$.

Long-term rate region: Let $\mathcal{V}$ be the system rate region, i.e., the set of all long-term feasible service rates under all possible feedback allocation policies. We characterize this set through the use of Static Service Split (SSS) scheduling rules following the approach pursued by Andrews et al. [7]. The rule can be described as follows. In channel state $\mathbf{m}$, the scheduler chooses bit allocation $\mathbf{b}$ with probability $\phi_{\mathbf{mb}}$; a SSS policy is completely characterized by a stochastic matrix $\mathbf{\Phi}$. The long-term rate region for this space of policies is written as

$$\mathcal{V} = \left\{ \boldsymbol{\nu}(\mathbf{\Phi}) : \sum_{b\in\mathcal{B}} \phi_{\mathbf{mb}} = 1,\ \phi_{\mathbf{mb}} \in [0,1], \forall \mathbf{m}, \mathbf{b} \right\}, \quad (1)$$

where $\boldsymbol{\nu}(\mathbf{\Phi}) = \sum_{\mathbf{m}\in\mathcal{M}} \pi_{\mathbf{m}} \sum_{\mathbf{b}\in\mathcal{B}} \phi_{\mathbf{mb}} \boldsymbol{\mu}(\mathbf{m}, \mathbf{b})$ and $\boldsymbol{\mu}(\mathbf{m}, \mathbf{b}) = [\mu_1(m_1, b_1)\ \mu_2(m_2, b_2) \ldots \mu_K(m_K, b_K)]^T$; $\boldsymbol{\nu}(\mathbf{\Phi})$ is the long-term average rate under scheduling policy $\mathbf{\Phi}$ since $\sum_{\mathbf{b}\in\mathcal{B}} \phi_{\mathbf{mb}} \boldsymbol{\mu}(\mathbf{m}, \mathbf{b})$ represents the expected rate while in channel state m, which is subsequently averaged over all channel states. In the following section, we comment on why it is sufficient to consider SSS feedback allocation policies in order to characterize the system rate region in the context of specific long-term system objectives that were briefly introduced in Section I.

III. Long-term network objectives

In Sections III.A and III.B, we define the two objectives that we briefly introduced earlier and justify the use of SSS policies to characterize the system rate region for each objective. The aim of this section is to establish that it is sufficient to solve an online weighted sum-rate maximization problem in order to achieve either long-term objective. This allows us to propose an optimal feedback allocation algorithm in Section IV, which solves this weighted sum-rate maximization problem at every scheduling instant. Theorem 2, which is a direct generalization of Theorem 1 constitutes a new contribution, one that paves the way for the development of a reduced-complexity approximation algorithm in Section V for applications that demand faster running times.

A. Queue stability

Assume that each user $k,\ k = 1, 2, \ldots, K$, has a queue of untransmitted packets with queue-length $Q_k[t]$ and associated arrival rate λ_k. The state of the system at time t is given by $\mathbf{S}[t] = \{\mathbf{m}[t], \mathbf{Q}[t]\}$. A mapping H from the state $\mathbf{S}[t]$ to a probability distribution $H(\mathbf{S}[t])$ on the set of queues $\{1, 2, \ldots, K\}$ is called a scheduling policy. This means that when the system is in state $\mathbf{S}[t]$, user k is picked for service according to the probability distribution $H(\mathbf{S}[t])$. Let $A_k[t]$ denote the packet arrival process for user k. For simplicity, let us assume that $A_k[t]$ is an ergodic Markov chain and that the arrival processes are mutually independent across users. Under these standard assumptions, the queue-state process is Markov and evolves according to

$$\mathbf{Q}[t] = \mathbf{Q}[t-1] + \mathbf{A}[t] - \mathbf{D}[t],$$

where $D_k[t] = \min\{Q_k[t], \mu_k(m[t], b^*[t])\}$; $b^*[t]$ is the allocation decision at time t. *Queue stability* is traditionally defined as the positive recurrence of the queue-state process $\mathbf{Q}[t]$ under a given scheduling policy.

The following theorem forges the connection between generic scheduling policies H and the space of SSS policies in the context of stability. It states that if some feedback allocation policy (possibly randomized, history-dependent, etc.) can stabilize a system, then there exists a SSS policy, as given in (1), that can also stabilize the system. In particular, the theorem says that one can obtain a throughput-optimal feedback allocation strategy by solving a linear program. We refer the reader to Andrews et al. [7] for the proof where the authors prove the claim under a definition of scheduling policies that maps the state $\mathbf{S}[t]$ to a probability distribution on the users indices $\{1, \ldots, K\}$ as opposed to a probability distribution on the set of bit allocations $\mathcal{B}$. The core idea of the proof involves a marginalization across the queue states $\mathbf{q}[t]$ in order to compute an equivalent SSS probability that picks an allocation or user in a given channel state $\mathbf{m}[t]$.

Theorem 1. *If a scheduling rule H exists under which the system is stable, then there exists a SSS scheduling policy $\mathbf{\Phi}$ such that the system is stable, i.e., $\boldsymbol{\lambda} < \boldsymbol{\nu}(\mathbf{\Phi})$.*

□

This theorem, in particular, justifies our use of SSS policies in the previous Section in order to characterize the rate region or stability region[3], equivalently, of an unsaturated system. The above theorem directly motivates the computation of a stabilizing SSS policy Φ^* given arrival rate vector $\boldsymbol{\lambda}$, through the following linear program

$$\begin{array}{rcll} \mathbf{\Phi}^* & = & \text{argmin} & c \\ & & \text{s.t} & \boldsymbol{\lambda} \leq c\boldsymbol{\nu}(\mathbf{\Phi}) \\ & & & \sum_{\mathbf{b}\in\mathcal{B}} \phi_{\mathbf{mb}} = 1,\ \forall m \in \mathcal{M} \\ & & & \phi_{\mathbf{mb}} \in [0,1], \forall \mathbf{m}, \mathbf{b} \end{array} . \quad (2)$$

Unfortunately, the linear program (2) is difficult to solve owing to the fact that the stochastic matrices $\mathbf{\Phi}$ have dimension $|\mathcal{M}| \times |\mathcal{B}| = M \times \binom{B+K-1}{K-1}$. Furthermore, we reiterate that the scheduler would require apriori knowledge of the arrival rates in order to perform this computation.

To alleviate this requirement on apriori knowledge of arrival rates, Tassiulus and Ephremedis [8] proposed the well-known *max-weight* or *back-pressure* online scheduling

[3] The stability region of an unsaturated system is defined as the set of arrival rates $\Lambda \subset \mathbb{R}_+^K$ that are stabilizable over the entire space of scheduling policies.

algorithm. Observing the natural connection between the independent sets defined by Tassiulus and Ephremedis in [8] and the feedback bit allocations in our model, it follows that if $\boldsymbol{\lambda} < \boldsymbol{\nu}(\bar{\boldsymbol{\phi}})$ for some SSS scheduling matrix $\bar{\boldsymbol{\phi}}$, then the following per-instant scheduling rule

$$\mathbf{b}^*[t] = \text{argmax }_{\mathbf{b}\in\mathcal{B}}\mathbf{Q}[t]^T\boldsymbol{\mu}(\mathbf{m}[t],\mathbf{b}) \tag{3}$$

stabilizes the system.

We give here a direct generalization of the above result in the following theorem, which essentially states that by calculating a β-approximate solution, $\beta \in [0,1]$ to (3) in every time slot, one can achieve a β-fraction of the stability region $\mathcal{V}$. This becomes important in the sequel, when we consider efficient but approximate algorithms for stability.

Theorem 2. *If $\boldsymbol{\lambda} < \beta\boldsymbol{\nu}(\bar{\boldsymbol{\phi}}), \beta \in (0,1]$ for some SSS scheduling matrix $\bar{\boldsymbol{\phi}}$, then a β-approximation to the following per-instant scheduling rule*

$$\mathbf{b}^*[t] = \textit{argmax }_{\mathbf{b}\in\mathcal{B}}\mathbf{Q}[t]^T\boldsymbol{\mu}(\mathbf{m}[t],\mathbf{b}) \tag{4}$$

stabilizes the system.

□

B. Utility maximization

The following alternate long-term network objective, proposed in [9], is applicable to saturated systems where each user has an infinite amount of data to be served (transmitted). For such systems, the state is given by $\mathbf{S}[t] = \mathbf{m}[t]$ and hence, any scheduling rule is automatically an SSS scheduling rule thereby justifying our earlier characterization of rate region $\mathcal{V}$ in (1). In such systems. we are concerned with optimizing the vector of long-term service rates $\boldsymbol{\nu}(\boldsymbol{\phi})$ such that we maximize some utility function $H(\boldsymbol{\nu})$ over the region $\mathcal{V}$ introduced earlier, i.e., we are interested in

$$\text{maximize}_{\boldsymbol{\nu}\in\mathcal{V}} \quad H(\boldsymbol{\nu}). \tag{5}$$

The following two classes of long-term utility functions are defined in [9]:
(i) Type I Utility Function - $H(\mathbf{u})$ is a continuous strictly concave function on $\mathbb{R}_+^K$. In addition, $H(\mathbf{u})$ is continuously differentiable, i.e., the gradient ∇H is finite and continuous everywhere in $\mathbb{R}_+^K$, ex. $H(\mathbf{u}) = \sum_{k=1}^K c_k u_k$.
(ii) Type II Utility Function - $H(u) = \sum_{k=1}^K H(u_k)$ where each $H(u_k)$ is a strictly concave continuously differentiable function, defined for all $u_k > 0$ and such that $H(u_k) \to -\infty$ as $u_k \to 0$, ex. $H(\mathbf{u}) = \sum_{k=1}^K \log(u_k)$.
For the aforementioned utility functions, [9] shows that the following gradient-weighted sum-rate maximization at each instant

$$\mathbf{b}^*[t] = \text{argmax }_{\mathbf{b}\in\mathcal{B}}\nabla H\left(\boldsymbol{\mu}_{emp}^{\delta}[t]\right)^T\boldsymbol{\mu}(\mathbf{m}[t],\mathbf{b}) \tag{6}$$

where

$$\boldsymbol{\mu}_{emp}^{\delta}[t] = (1-\delta)\boldsymbol{\mu}_{emp}^{\delta}[t] + \delta\boldsymbol{\mu}(\mathbf{m}[t],\mathbf{b}^*[t])$$

is the empirical rate vector measured till time t solves (5) for δ sufficiently small. Formally stated, the statement proven in [Theorem 2, [9]] says:

Theorem 3. *Let $\mathcal{A}$ be a bounded subset of $\mathbb{R}_+^K$. Then, for any $\varepsilon > 0$, there exists $T > 0$ (depending on ε and $\mathcal{A}$) such that*

$$\lim_{\delta\to 0}\sup_{\boldsymbol{\mu}_{emp}^{\delta}[0]\in\mathcal{A}, t>\frac{T}{\delta}} P\left(||\boldsymbol{\mu}_{emp}^{\delta}[t] - \boldsymbol{\nu}^*|| > \varepsilon\right) = 0$$

□

IV. Optimal allocation through dynamic programming

In Section III, we have established that for queue stability in (4) and for Type I/II utility maximization in (6), we are interested in the following online weighted sum-rate maximization problem

$$\text{maximize}_{\mathbf{b}\in\mathcal{B}} \quad \mathbf{w}^T\boldsymbol{\mu}(\mathbf{m}[t],\mathbf{b}), \tag{7}$$

where $\mathbf{w} = [w_1,\ldots,w_K]^T$ is a vector of non-negative weights. Herein, the focus of this paper becomes algorithmic in that we propose novel solutions to (7) in Theorems 4-7 that explore the natural trade-off between accuracy and complexity. Theorem 4 is a first step in this direction, the proof of which has been omitted due to lack of space.

Theorem 4. *The online resource allocation problem (7) can be solved using dynamic programming incurring complexity $\mathcal{O}\left(KB^2\right)$.*

□

V. Reduced-complexity resource allocation

Thus far, we have identified the optimal online allocation problem (7) in order to achieve either long-term objective and we have proposed an exact solution using dynamic programming, which has complexity $\mathcal{O}\left(KB^2\right)$. While this pseudo-polynomial[4] complexity might not be too large for most applications considering that it would take $\mathcal{O}(B)$ for the base station to write these bits, some applications might demand faster algorithms. It is also crucial to recognize that once computed, communicating the optimal feedback allocation back to the users would incur an overhead of $\log_2\binom{B+K-1}{K-1}$ bits since the base-station needs to potentially communicate $|\mathcal{B}| = \binom{B+K-1}{K-1}$ messages. Through the remainder of this section, we consider an uplink scenario where all nodes (including the base-station) are equipped with multiple antennas and the adopted transceiver scheme is single-stream beamforming and combining[5]. We show that for this choice of physical layer signalling protocol, which directly impacts the structure of set $\mathcal{U}(m,b)$, the weighted-sum-rate maximization problem in (7) takes on a specific form that allows for the development of an approximation algorithm with significantly reduced complexity $\mathcal{O}(K\log_2 K)$. We begin this section by investigating the effects of limited feedback on the aforementioned class of MIMO systems.

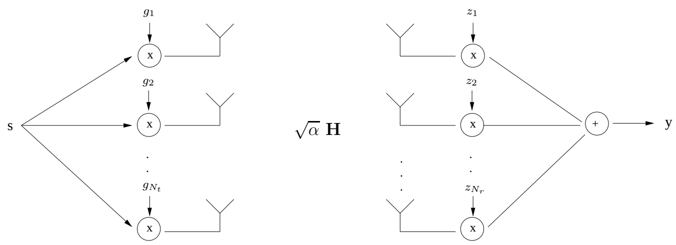

Fig. 2. Single-stream beamforming and combining MIMO system.

A. Single-stream MIMO with limited feedback

The classical $N_t \times N_r$ single-stream beamforming and combining MIMO link for a typical user (shown in Fig. 2) can be described using the following received signal model,

$$y = \sqrt{\alpha}\mathbf{z}^{\dagger}\mathbf{H}\mathbf{g}s + \mathbf{z}^{\dagger}\mathbf{n}, \tag{8}$$

where

$\mathbf{s}$			$\sim$	Complex Gaussian transmit codeword with $\mathbb{E}[\lvert s\rvert^2] = P$
$\mathbf{n}$	$\in$	$\mathbb{C}^{N_r}$	$\sim$	$\mathcal{CN}(0, N_o\mathbf{I})$ is additive white Gaussian noise
$\mathbf{g}$	$\in$	$\mathbb{C}^{N_t}$	:	Transmit beamformer with $\lvert\lvert\mathbf{g}\rvert\rvert^2 = 1$ to satisfy the transmit power constraint
$\mathbf{z}$	$\in$	$\mathbb{C}^{N_r}$	:	Receive combiner
$\mathbf{H}$	$\in$	$\mathbb{C}^{N_r \times N_t}$	:	Complex-valued MIMO channel

The model in (8) is a comprehensive description of the wireless channel in that it explicitly accounts for the composite effects of small-scale (SS) fading and large-scale (LS) fading. We use α to represent the path-loss or shadowing effects of the channel, henceforth referred to as *LS effects*, while the matrix $\mathbf{H}$ denotes SS fading. Composite models have been used in past literature (see [15] and references therein). The signal-to-interference-plus-noise ratio (SINR) for this system can be written as

$$\text{SINR} = \frac{|\mathbf{z}^{\dagger}\mathbf{H}\mathbf{g}|^2}{||\mathbf{z}||^2}\frac{P\alpha}{N_o}. \tag{9}$$

It is well-known that the SINR in (9) can be maximized by setting $\mathbf{g}^* = \mathbf{v}$ and $\mathbf{z}^* = \mathbf{H}\mathbf{g}^*$ where $\mathbf{v}$ is the right singular vector corresponding to the maximum singular value σ of the channel matrix $\mathbf{H}$. By introducing user indices, the maximum SINR for user k can be written as

$$\text{SINR}_{k,PF} = \frac{\alpha_k P_k \sigma_k^2}{N_o}. \tag{10}$$

The choice of notation reflects the fact that the user requires perfect feedback of the right singular vector $\mathbf{v}_k$ from the base-station in order to achieve this maximum SINR. However, feedback in realistic systems is imperfect due to limited feedback budgets, the primary motivation for this work. Through the remainder of this section, we restrict our attention to quantization error; error that is introduced when the base-station quantizes the optimal precoder $\mathbf{v}_k$ using b_k bits in preparation for feedback[6]. Following literature (see ex. [11]), the feedback link is assumed to be delay- and error-free. We assume that user k uses a quantized beamformer $\mathbf{v}_{q,k}$ that can be modeled as

$$\mathbf{v}_{q,k} = \mathbf{v}_k + \mathbf{e}_k. \tag{11}$$

Here, $\mathbf{e}_k$ is an additive error term which represents the uncertainty caused due to quantization at the base-station. We assume that this error comes from a deterministic set that is bounded, i.e., $||\mathbf{e}_k||^2 \leq \Delta(b_k)$, where $\Delta(b_k)$ in an invertible non-increasing function of b_k such that $\Delta(b_k) \to 0$ as $b_k \to \infty$. Such norm-bounded additive error models have been used in the past [12]. Based on the above definitions, by substituting (11) in (8), we can write the SINR with imperfect feedback as

$$\text{SINR}_{k,IF} = \frac{\alpha_k P_k \sigma_k^2}{\alpha_k P_k \sigma_k^2 |\mathbf{v}_k^{\dagger}\mathbf{e}_k|^2 + N_o}. \tag{12}$$

The additional interference term in the denominator of (12) represents the degradation due to quantization error.

B. Time-scales and structure of set $\mathcal{U}(m, b)$

In this section, we describe the structure of set $\mathcal{U}(m, b)$ that arises out of employing the single-stream MIMO physical layer scheme described earlier.

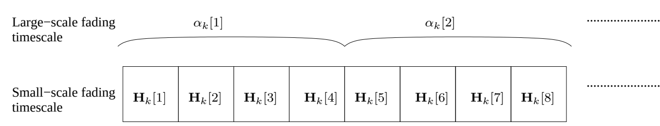

Fig. 3. Composite effects of small-scale fading and large-scale fading in a wireless channel with $D = 4$.

We consider making feedback allocations once every LS fading coherence time, which typically spans mutiple SS fading coherence times, say D of them, as shown in Fig. 3. Such a design choice has two benefits; first, it might require too much overhead to compute and communicate optimal allocations on the SS fading time-scale, which typically spans a few milliseconds. Second, this allows each user to estimate their LS coefficient α_k without the need for feedback from the base-station by exploiting reciprocity on the downlink. This is possible since path-loss and/or shadowing are dependent solely on the distance between the user and the base-station. The increasing availability of GPS-enabled devices also offers the user an alternate means to compute their path-loss.

Capturing the two separate time-scales, we define the channel state as

$$\mathbf{m}[t] = \{\alpha[t], [\mathbf{H}_k[(t-1)D+1], \ldots, \mathbf{H}_k[tD]]\,,\; k = 1, \ldots, K\}$$

for the single-stream MIMO system we are considering. We assume that $\{\alpha[t]\}$, is a finite-state process that is either *(i)* i.i.d. across time or *(ii)* an ergodic Markov chain[7], taking

[4]An algorithm has pseudo-polynomial complexity if its running time is a polynomial in the size of the input in unary. The size of the input to (7) in unary at most $KBA_{max} + B = \mathcal{O}(KB)$ where $A_{max} = \max_{(i,j)} A(i,j)$.

[5]Single-stream beamforming and combining multiple-input-multiple-output (MIMO) systems have been extensively studied in the past [4], [18], [19] and are an attractive method for achieving reliable data transmission through significant diversity and array gain

[6]The typical quantizer [4] would create a Voronoi partition of the unit sphere with 2^{b_k} cells.

[7]Markovian and i.i.d. models for user mobility in a cell (and hence path-loss) have been utilized by El Gamal et al. [13] and Toumpis et al. [14] respectively in studying how mobility impacts the performance of a wireless network.

values from the set $\mathcal{P}$ with a unique stationary distribution $\{\pi_\alpha\}_{\alpha\in\mathcal{P}}$. On the faster time-scale, we assume that $\{[\mathbf{H}_k[(t-1)D+1],\ldots,\mathbf{H}_k[tD]]\,,\ k=1,\ldots,K\}$ is again a finite-state process that is either i.i.d. across time or ergodic Markov taking values from the set $\mathcal{H}$. Traditionally, each element of the channel matrix $\mathbf{H}_k$ is modeled as a complex Gaussian random variable. However, since we are interested in finite-state processes, one can discretize this random variable and create set $\mathcal{H}$ by sampling the support of its probability density function sufficiently finely. As is the case in past literature (see [15] and references therein), large-scale fading is assumed to be independent of the small-scale fading.

In each state $m\in\mathcal{M}=\mathcal{P}\times\mathcal{H}$, given bit allocation b, we assume that user k transmits at rate $\mu_k(\alpha_k,b_k)$ independent of $\{[\mathbf{H}_k[(t-1)D+1],\ldots,\mathbf{H}_k[tD]]\,,\ k=1,\ldots,K\}$, i.e., uncertainty set $\mathcal{U}_k(m_k,b_k)$ is a singleton. Systems that choose to transmit at a fixed rate $\mu_k(\alpha_k,b_k)$ during the course of an entire LS coherence time would immediately be susceptible to outages or packet drops. This is because a particular SS fading realization within the larger coherence time might not be able to support the chosen transmission rate in accordance with Shannon's capacity formula. Such a transmission scheme would fall under the widely-pursued outage capacity [17] framework where the transmitter chooses a fixed rate for an extended length of time while allowing for a small outage probability. In this framework, outages arise due to delay constraints that dictate that a packet must be decoded within a SS coherence time.

Given a fixed α_k and bit allocation b_k through the course of a large coherence time, we define $\mu_k(\alpha_k,b_k)$ to be the *goodput* (a notion that is discussed by Lau et al. [16]) when transmitting at the maximum possible rate $\gamma_k^*(\alpha_k,b_k)$ while allowing for an outage probability of at most ϵ_k, i.e.,

$$\mu_k(\alpha_k,b_k)\triangleq\gamma_k^*(\alpha_k,b_k)(1-\epsilon_k)$$

To compute $\gamma_k^*(\alpha_k,b_k)$, we need to quantify the outage probabilty of the single-stream beamforming/combining MIMO system. From (12), the SINR with imperfect feedback is a random variable whose distribution depends on the distribution of σ_k^2 and $\mathbf{v}_k$. Thus, the outage probability for user k that transmits at rate $\gamma_k(\alpha_k,b_k)$ can be written as

$$\mathbb{P}_{\sigma_k^2,\mathbf{v}_k}\left(\frac{\alpha_kP_k\sigma_k^2}{\alpha_kP_k\sigma_k^2|\mathbf{v}_k^\dagger\mathbf{e}_k|^2+N_o}\le 2^{\gamma_k(\alpha_k,b_k)}-1\right)\qquad(13)$$

In the interest of having (13) reflect an explicit dependence on the feedback allocation b_k through the uncertainty function $\Delta(b_k)$, which will allow us to proceed further with this computation, we form the following lower-bound

$$\begin{aligned}\mathrm{SINR}_{k,IF}&=\frac{\alpha_kP_k\sigma_k^2}{\alpha_kP_k\sigma_k^2|\mathbf{v}_k^\dagger\mathbf{e}_k|+N_o}\\&\ge\frac{\alpha_kP_k\sigma_k^2}{\alpha_kP_k\sigma_k^2||\mathbf{v}_k||^2||\mathbf{e}_k||^2+N_o}\\&\ge\frac{\alpha_kP_k\sigma_k^2}{\alpha_kP_k\sigma_k^2\Delta(b_k)+N_o}\triangleq\mathrm{SINR}_{k,IF}^{'}.\end{aligned}\qquad(14)$$

using the Cauchy-Schwartz inequality. It is clear that

$$\mathbb{P}_{\sigma_k^2,\mathbf{v}_k}(\mathrm{SINR}_{k,IF}\le 2^{\gamma_k(\alpha_k,b_k)}-1)\ge\mathbb{P}_{\sigma_k^2}(\mathrm{SINR}_{k,IF}^{'}\le 2^{\gamma_k(\alpha_k,b_k)}-1)$$

and therefore, by working with $\mathbb{P}_{\sigma_k^2}(\mathrm{SINR}_{k,IF}^{'}\le 2^{\gamma_k(\alpha_k,b_k)}-1)\le\varepsilon_k$ as our definition of outage probability, we are being conservative. We enforce the maximum outage probability constraint of ε_k and explicitly compute $\gamma_k^*(\alpha_k,b_k)$ as

$$\begin{aligned}&\mathbb{P}_{\sigma_k^2}(\mathrm{SINR}_{k,IF}^{'}\le 2^{\gamma_k(\alpha_k,b_k)}-1)\le\varepsilon_k\\ \Rightarrow\ &\frac{(2^{\gamma_k(\alpha_k,b_k)}-1)N_o}{\alpha_kP_k\left(1-(2^{\gamma_k(\alpha_k,b_k)}-1)\Delta(b_k)\right)}\le F_{\sigma^2}^{-1}(\varepsilon_k)\\ \Rightarrow\ &\gamma_k^*(\alpha_k,b_k)=\log_2\left(1+\frac{a_k}{1+a_k\Delta(b_k)}\right),\end{aligned}$$

where $a_k=\frac{\alpha_kP_kF_{\sigma_k^2}^{-1}(\varepsilon_k)}{N_o}$ and $F_{\sigma_k^2}(x)$ denotes the cumulative distribution function of σ_k^2. $\gamma_k^*(\alpha_k,b_k)$, as noted before, represents the maximum possible transmission rate that obeys the outage constraints.

Thus, we have computed the goodput when transmitting at $\gamma_k^*(\alpha_k,b_k)$ while incurring outage probability of at most ϵ_k as

$$\mu_k(\alpha_k,b_k)\triangleq\log_2\left(1+\frac{a_k}{1+a_k\Delta(b_k)}\right)(1-\epsilon_k).$$

From (1), the rate region for a system that employs the single-stream MIMO physical layer structure described thus far can be expressed in terms of

$$\begin{aligned}\boldsymbol{\nu}(\boldsymbol{\Phi})&=\textstyle\sum_{\mathbf{m}\in\mathcal{M}}\pi_{\mathbf{m}}\sum_{\mathbf{b}\in\mathcal{B}}\phi_{\mathbf{mb}}\boldsymbol{\mu}(\mathbf{m},\mathbf{b})\\&=\textstyle\sum_{\boldsymbol{\alpha}\in\mathcal{P}}\pi_{\boldsymbol{\alpha}}\sum_{\mathbf{b}\in\mathcal{B}}\phi_{\boldsymbol{\alpha}\mathbf{b}}\boldsymbol{\mu}(\boldsymbol{\alpha},\mathbf{b})\\&=\textstyle\sum_{\boldsymbol{\alpha}\in\mathcal{P}}\pi_{\boldsymbol{\alpha}}\sum_{\mathbf{b}\in\mathcal{B}}\phi_{\boldsymbol{\alpha}\mathbf{b}}\Big[\log_2\left(1+\frac{a_1}{N_o+a_1\Delta(b_1)}\right)\\&\quad(1-\varepsilon_1),\ldots,\log_2\left(1+\frac{a_K}{N_o+a_K\Delta(b_K)}\right)(1-\varepsilon_K)\Big]^T\end{aligned}$$

and the optimization in (7) takes the specific form

$$\mathrm{maximize}_{\mathbf{b}\in\mathcal{B}}\quad\sum_{k=1}^K w_k(1-\varepsilon_k)\log_2\left(1+\frac{a_k}{N_o+a_k\Delta(b_k)}\right).\qquad(15)$$

We absorb the success probability $(1-\varepsilon_k)$ into weight w_k henceforth.

C. Relaxation and approximation guarantees

For the rest of the paper, we assume a specific form for the uncertainty function, namely, $\Delta(b)=\frac{1}{c_1b+c_2},c_1,c_2>0$. Summarizing what we have done beyond (7) in Section IV, we have introduced a specific MIMO physical layer communication protocol that *(i)* has a feedback overhead requirement of $\frac{\log_2\binom{B+K-1}{K-1}}{\text{LS coherence time}}$ bits/s and as we will see in Theorems 5-7 below, *(ii)* allows us to develop an approximation algorithm to (15) to be solved in closed-form, incurring a complexity of $\mathcal{O}(K\log_2 K)$ instead of $\mathcal{O}(KB^2)$, while providing an approximation guarantee of $\left(\frac{c_1+c_2}{2c_1+c_2}\right)$. The proofs have been omitted due to lack of space.

Theorem 5. *Consider the following continuous relaxation of (15):*

$$\mathbf{b}^*[t]=\mathrm{argmax}_{\sum_k b_k\le B,b_k\in\mathbb{R}_+}\sum_{k=1}^K w_k[t]log_2\left(1+\frac{a_k[t]}{1+a_k[t]\Delta(b_k)}\right).$$

The solution to this relaxation with uncertainty function $\Delta(b)=\frac{1}{c_1b+c_2},c_1,c_2>0$ is given in (16) where η^* is chosen such that $\sum_k b_k^*=B$.

□

Theorem 6. *Computing the solution in (16) incurs a complexity of $\mathcal{O}(Klog_2K)$.*

$$b_k^* = \left[\frac{1}{2}\sqrt{(c_1(2c_2(a_k+1)+a_k(a_k+1)+a_kc_1)) + 4c_1^2(a_k+1)\left(\frac{w_k a_k^2 c_1}{\eta^* \log 2} - (c_2 a_k(a_k+1)+a_k c_2 + a_k^2)\right)} - \frac{c_1(2c_2(a_k+1)+a_k(a_k+1)+a_kc_1)}{2}\right]^+ \tag{16}$$

□

Once we solve for b_k^*, we apply a floor operation in order to enforce the integer constraints, i.e., we set $b_{k,INT}^* = \lfloor b_k^* \rfloor$. This leads us to the task of quantifying loss due to integrality, which we address in Theorem 7 below.

Theorem 7. *The bit allocation obtained by relaxing integer constraints followed by flooring gives an approximation factor of* $\frac{c_1+c_2}{2c_1+c_2}$.

□

The results in Theorems 5-7 are applicable to single-stream MIMO systems where the norm-squared quantization error is accurately bounded by uncertainty function $\Delta(b) = \frac{1}{c_1 b + c_2}$. We briefly comment on our choice of this function. Past research by Mondal and Heath [4] is relevant to this discussion. In this work, they show that the expected loss in SNR due to feedback quantization using b_k bits for a single-stream beamforming/combining MIMO system is well-approximated by $\frac{P_k \alpha_k}{N_o}\left(E[\sigma_k^2] - N_r\right) 2^{-\frac{b_k}{N_t-1}}$. While these results are not directly applicable to our setup owing to the fact that their loss in SNR is averaged over SS fading, it still suggests that $\Delta(b) = 2^{-cb}, c > 0$ might be a reasonable choice for uncertainty if one were to accept the average loss as a good approximation of the instantaneous loss in SNR. This can be seen, roughly, through the following analysis

$$\begin{aligned}
& \text{SINR}_{k,PF} - \text{SINR}_{k,IF} \approx \frac{P_k\alpha_k}{N_o}\left(E[\sigma_k^2]-N_r\right)2^{-\frac{b_k}{N_t-1}} \\
\Rightarrow\ & \frac{P_k\alpha_k\sigma_k^2}{P_k\alpha_k\sigma_k^2\Delta(b_k)+N_o} \approx \frac{P_k\alpha_k\sigma_k^2}{N_o} - \frac{P_k\alpha_k}{N_o}\left(E[\sigma_k^2]-N_r\right)2^{-\frac{b_k}{N_t-1}} \\
\Rightarrow\ & P_k\alpha_k\sigma_k^2\Delta(b_k) + N_o \approx \frac{P_k\alpha_k\sigma_k^2}{\frac{P_k\alpha_k\sigma_k^2}{N_o} - \frac{P_k\alpha_k}{N_o}\left(E[\sigma_k^2]-N_r\right)2^{-\frac{b_k}{N_t-1}}} \\
\Rightarrow\ & \Delta(b_k) \approx \left(\frac{N_o}{P_k\alpha_k\sigma_k^2}\right)\left[\frac{\sigma_k^2}{\sigma_k^2 - \left(E[\sigma_k^2]-N_r\right)2^{-\frac{b_k}{N_t-1}}} - 1\right] \\
\Rightarrow\ & \Delta(b_k) \approx \left(\frac{N_o}{P_k\alpha_k\sigma_k^2}\right)\left[\frac{2^{\frac{b_k}{N_t-1}}\sigma_k^2 - 2^{\frac{b_k}{N_t-1}}\sigma_k^2 + \left(E[\sigma_k^2]-N_r\right)}{2^{\frac{b_k}{N_t-1}}\sigma_k^2 - \left(E[\sigma_k^2]-N_r\right)}\right] \\
\Rightarrow\ & \Delta(b_k) \approx \left(\frac{N_o}{P_k\alpha_k\sigma_k^2}\right)\frac{\left(E[\sigma_k^2]-N_r\right)}{2^{\frac{b_k}{N_t-1}}\sigma_k^2 - \left(E[\sigma_k^2]-N_r\right)} \\
\Rightarrow\ & \Delta(b_k) \approx \Theta(2^{-cb_k}) \text{ for some } c > 0.
\end{aligned}$$

This choice also agrees with common intuition since the number of quantization levels increases in the number of bits b as 2^b. However, such a choice would destroy the concavity of the objective function (7) thereby precluding the use of the KKT conditions as a tool for finding the optimal solution. We argue that by carefully picking constants c_1 and c_2, one can form an upper-bound to the function 2^{-cb} over the range of interest $b \in \{0, \ldots, B\}$. This conservative approach ensures that our performance goals are met while allowing us to exploit the benefits of convexity in solving (7). Consequently, we are able to study the behavior of the optimal allocation as a function of the system parameters.

In conclusion, Theorem 4 connects the performance of the $\left(\frac{c_1+c_2}{2c_1+c_2}\right)$-approximate online algorithm given in this section to the long-term stability region of the policy.

VI. Conclusion

In this paper, we propose an optimal feedback allocation policy for cellular uplink systems where the base station has a limited feedback budget. The optimality is in the sense of queue stability for unsaturated queueing regimes and long-term utility maximization for saturated queueing regimes. The optimal allocation policy involves solving a weighted sum-rate maximization problem at every scheduling instant. This problem is solved using dynamic programming incurring pseudo-polynomial complexity in the number of users and the total bit budget. For single-stream beamforming and combining MIMO physical layer communication schemes, we propose a relaxation to the optimal feedback allocation problem that can be solved in closed-form, incurring polynomimal complexity. We provide approximation guarantees for the proposed relaxation under a specific class of uncertainty functions.

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